MEAM 5360: VISCOUS FLUID FLOW

Assignment 2: Unsteady Flow in Finite Pipe

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Introduction

For this problem, we are looking at the unsteady flow through a finite pipe of length L. We are told that the inlet pressure is P_0, the outlet pressure is P_L, and the system starts at rest. At $t = 0$ our pressure gradient is introduced. We consider only the 1-dimensional time-dependent flow. This means out velocity only depends on one spatial variable, r i this case, in addition to time. We assume no slip boundary conditions at the pipe walls and a bounded solution everywhere else.

Over the course of this homework, we will be solving this problem. There will be several specific questions asked throughout which we will use to separate the sections.

Assumptions

1. Incompressible
2. Newtonian
3. $v_r = 0$
4. $v_\theta = 0$
5. $v_z = v_z(r,t) = 0$

Question 1 - Combination of Variables

We cannot use combination of variables for this problem since it does not give a non-zero velocity profiles as time goes to infinity. However, since we know physically that this sort of pipe flow will have a non-zero velocity profile after the initial condition. This means that our solution cannot be found with combination of variables, so we instead go to separation of variables.

Question 2 - Non-dimensionalization

To begin, we will first simplify our dimensional equation. Once we have simplified, we will introduce the appropriate non-dimensionalization and see how the math works out. We will start with the continuity equation in cylindrical coordinates.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{1}$$

Using assumptions 1, 3, and 4, the first three terms go to zero, respectively, which leaves us with

$$\frac{\partial v_z}{\partial z} = 0 \tag{2}$$

In analyzing the momentum equation, it's notable that velocity components in the θ and r directions are absent. Consequently, all terms in the corresponding momentum equations for these directions equate to zero, leaving only the pressure gradient term. This observation suggests that pressure remains constant in both the θ and r directions, directing our attention towards the z-momentum equation.

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial \theta^2}\right] + \rho g_z \tag{3}$$

On the left hand side, the second and third terms go to zero through assumption 3 and 4 whilst the fourth term goes to zero via the continuity result in Eq. (2). However, we need to not that the first term is not zero because the flow is unsteady. On the right hand side, the last two terms are equal to zero through assumption 5 which suggests that velocity is only a function r and t and the continuity result. This leaves us with the following equation

$$\rho\frac{dv_z}{dt} = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(\frac{r\partial v_z}{\partial r}\right) \tag{4}$$

Now we can non-dimensionalize the variables with ϕ as the non-dim velocity, ζ as non-dim radius and tau as non-dim time. We will start with the velocity

$$\phi = \frac{v_z}{\frac{(P_0 - P_L)R^2}{4\mu L}} \tag{5}$$

$$v_z = \frac{(P_0 - P_L)R^2\phi}{4\mu L} \tag{6}$$

Following this, we can proceed similarly with regards to the radial dimension, denoted as r

$$\zeta = \frac{r}{R} \tag{7}$$

$$r = \zeta R \tag{8}$$

Lastly, we can extend this analysis to include the time variable.

$$\tau = \frac{t}{\frac{\rho R^2}{\mu}} \tag{9}$$

$$t = \frac{\tau \rho R^2}{\mu} \tag{10}$$

Next, we plug these dimensioned variables in terms of their non-dimensionalized form into Eq. (4).

$$\frac{\partial}{\partial r} = \frac{1}{R}\frac{\partial}{\partial \zeta}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \left(\frac{\tau\rho R^2}{\mu}\right)} = \frac{\mu}{\rho R^2}\frac{\partial}{\partial \tau}$$

Since we know that our pressure gradient is from high to low we can say that

$$\frac{\partial P}{\partial z} = \frac{P_L - P_0}{L} \tag{11}$$

When we put this back into the main equation, there is a negative sign which makes it $P_0 - P_L$.

$$\rho \frac{\mu}{\rho R^2}\frac{\partial}{\partial \tau}\left(\frac{(P_0 - P_L)R^2\phi}{4\mu L}\right) = \frac{P_0 - P_L}{L} + \frac{\mu}{\xi R}\frac{1}{R}\frac{\partial}{\partial \xi}\left(\xi R \frac{1}{R}\frac{\partial}{\partial \xi}\left(\frac{(P_0 - P_L)R^2\phi}{4\mu L}\right)\right)$$

As a result, we arrive at the simplified non-dimensional momentum equation.

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) \tag{12}$$

Regarding the boundary conditions, we consider the velocities $v_z(R,t)$ and $v_z(r,t)$ at the wall ($\xi = R/R = 1$) and center of the pipe ($r = 0$, $\xi = 0/R = 0$) respectively. At the wall, adhering to the no-slip condition, the velocity equals zero, mirroring that of the wall. Hence, for the first boundary condition, we express $\phi(\xi = 1, t) = 0$. Examining the second boundary condition, the velocity at the pipe's center must be finite, indicating a non-zero value at $r = 0$, $\xi = 0/R = 0$. This yields $\phi(\xi = 0, t) = finite$. Finally, the initial condition stipulates that the fluid is initially at rest, implying zero velocity at $t = 0$ so $\tau = 0$ from Eq. (9). Thus, $\phi(r, \tau = 0) = 0$.

Question 3 - Steady State Solution

Now that we have the governing differential equation, we wish to find the solution. To do this, we introduce the following decomposition:

$$\phi = \phi_\infty(\xi) - \phi_\tau(\xi, \tau) \tag{13}$$

Substituting this into our governing equation we get:

$$\frac{\partial(\phi_\infty - \phi_\tau)}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial(\phi_\infty - \phi_\tau)}{\partial \xi}\right) \tag{14}$$

By definition, ϕ_τ decays to zero as τ goes to infinity. If we take the limit of the above equation as τ approaches infinity, and noticing ϕ_∞ is only a function of ξ, we can reduce the problem to:

$$0 = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_\infty}{\partial \xi}\right) \tag{15}$$

This is a simple ODE which can be solved through integration. Notice that we require both the no-slip and bounded boundary conditions to hold in this case, but do not use the initial condition. This we can say $\phi_\infty(0) < \infty, \phi_\infty(1) = 0$. Below is the work for the final solution.

$$-4\xi = \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_\infty}{\partial \xi}\right)$$

$$-2\xi^2 + C_1 = \xi \frac{\partial \phi_\infty}{\partial \xi}$$

$$-2\xi + \frac{C_1}{\xi} = \frac{\partial \phi_\infty}{\partial \xi}$$

$$\phi_\infty = -\xi^2 + C_1 \ln(\xi) + C_2 \tag{16}$$

Eq. (16) is the general form of the solution. If we require the solution stays bounded at 0, then it is clear to see that $C_1 = 0$. If we then plug in $\xi = 1$, we can see the $C_2 = 1$. Given this, we get the following result for the steady-state solution:

$$\phi_\infty = 1 - \xi^2 \tag{17}$$

This is our non-dimensional solution. Recall the non-dimensionalization we used in Eq. (5):

$$\phi = \frac{4\mu L V_z}{(P_0 - P_L)R^2}$$

However, for an infinitely long pipe, our pressure gradient term must be written as a derivative. That is to say:

$$\phi = \frac{4\mu V_z}{-\frac{\partial P}{\partial z} R^2} \tag{18}$$

Given this, an the fact that $\xi = \frac{r}{R}$, we can dimensionalize our steady-state solution as follows:

$$\frac{4\mu V_z}{-\frac{\partial P}{\partial z} R^2} = 1 - \left(\frac{r}{R}\right)^2$$

$$V_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}\left(r^2 - R^2\right) \tag{19}$$

This gives us the velocity profile for a Newtonian, Incompressible Fluid in an infinitely long pipe.

Question 4 - Transient Solution

Now that we have the steady-state solution, we need to solve the transient case. Recall Eq. (14):

$$\frac{\partial(\phi_\infty - \phi_\tau)}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial(\phi_\infty - \phi_\tau)}{\partial \xi}\right)$$

If we use the properties of derivative to separate this, we can see this can be written as:

$$\frac{\partial \phi_\infty}{\partial \tau} - \frac{\partial \phi_\tau}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_\infty}{\partial \xi}\right) - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_\tau}{\partial \xi}\right) \tag{20}$$

In the steady state solution, we found that the first to terms on the right-hand side together become zero. In addition, we know that the time derivative of the steady-state solution is zero. Together, this simplifies our equation to:

$$\frac{\partial \phi_\tau}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi_\tau}{\partial \xi}\right) \tag{21}$$

To solve this, we introduce the separation of variables. We assume that our solution is the product of two functions, one for each variable. That is to say:

$$\phi_\tau = \Xi(\xi)T(\tau) \tag{22}$$

If we substitute this form into our differential equation and rearrange, we can get:

$$\frac{\partial(\Xi T)}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial(\Xi T)}{\partial \xi}\right)$$

$$\frac{1}{T}\frac{\partial T}{\partial \tau} = \frac{1}{\Xi}\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \Xi}{\partial \xi}\right) \tag{23}$$

From this equation, since the left is only a function of τ and the right is only a function of ξ, we know that both sides will be equal to a constant we will call $-\alpha^2$. The negative is imperative for this problem. This variable can take on any complex value. Given this, we can get two ODEs as follows:

$$\frac{\partial T}{\partial \tau} + \alpha^2 T = 0 \tag{24}$$

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \Xi}{\partial \xi}\right) + \alpha^2 \Xi = 0 \tag{25}$$

Tackling the time equation first, it is clear to see that this has solution:

$$T = C_0 e^{-\alpha^2 \tau} \tag{26}$$

In addition, we are told that the spatial differential equation has the following solution:

$$\Xi = C_1 J_0(\alpha\xi) + C_2 Y_0(\alpha\xi) \tag{27}$$

From here, notice that we require our solution to remain bounded at $\xi = 0$. Since $Y_0(0)$ goes to infinity, this condition requires $C_2 = 0$. In addition, we require the no-slip boundary condition to hold, which requires:

$$0 = C_1 J_0(\alpha) \tag{28}$$

If we say $C_1 = 0$, we would arrive at the trivial solution of $\phi = 0$, which is not physical. Instead, we require that $J_0(\alpha) = 0$. Since J_0 has infinitely many zeros, we will label them with the subscript n. That is to say, $J_0(\alpha_n) = 0$ where α_n is the nth zero of the Bessel Function. This simplifies our spatial solution to:

$$\Xi = \sum_{n=1}^{\infty} C_{1,n} J_0(\alpha_n \xi) \tag{29}$$

To enforce the initial condition, we first need to put everything together. Putting our two solutions together, we get:

$$\phi_\tau = \sum_{n=1}^{\infty} B_n J_0(\alpha_n \xi) e^{-\alpha_n^2 \tau} \tag{30}$$

Here, we will consider what the Bessel Functions are and what they do for us here. As can be seen, the Bessel Functions are the basis functions that provide us solutions when solving problems in Cylindrical coordinates. They provide the proper shaping for cylindrical coordinates. This helps in providing a closed-form solution for cylindrical problems.

Question 5 - Initial Condition and Results

Notice that here we have combined our two unknowns into one unknown by $B_n = C_0 C_{1,n}$. This is fine since these were unknown values, so their product is simply one new unknown instead of two unknowns. Our initial condition states:

$$\phi(\xi, 0) = \phi_\infty(\xi) - \phi_\tau(\xi, 0) = 0$$

Rearranging, this initial condition requires

$$1 - \xi^2 = \sum_{n=1}^{\infty} B_n J_0(\alpha_n \xi) \tag{31}$$

To find all the values of B_n that make this relationship true, we take advantage of the Orthogonality of the Bessel Functions. We are provided the following relationships:

$$\int_0^1 \xi J_0(\alpha_n \xi) J_0(\alpha_m \xi) d\xi = 0 \text{ if n} \neq \text{m} \tag{R1}$$

$$\int_0^1 \xi J_0(\alpha_n \xi) J_0(\alpha_m \xi) d\xi = \frac{J_1^2(\alpha_n)}{2} \text{ if n=m} \tag{R2}$$

$$\frac{d}{d\xi}\left[\xi^p J_p(\alpha \xi)\right] = \alpha \xi^p J_{p-1}(\alpha \xi) \tag{R3}$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x) \tag{R4}$$

From here, we can multiply both sides of Eq. (31) by $\xi J_0(\alpha_m \xi)$ and integrate along our domain to arrive at:

$$\int_0^1 (1 - \xi^2) \xi J_0(\alpha_m) d\xi = \int_0^1 \sum_{n=1}^{\infty} B_n \xi J_0(\alpha_n \xi) J_0(\alpha_m \xi) d\xi \tag{32}$$

Since the right hand side goes to zero if $n \neq m$, we can assume $n = m$ and simplify this to:

$$\int_0^1 (1 - \xi^2) \xi J_0(\alpha_n) d\xi = \int_0^1 \sum_{n=1}^{\infty} B_n \xi J_0(\alpha_n \xi) J_0(\alpha_n \xi) d\xi \tag{33}$$

We can also flip the order of the integration and the sum. This means that our right-hand side takes the form of Eq. (R2) and we can say:

$$\int_0^1 (1 - \xi^2) \xi J_0(\alpha_m \xi) d\xi = B_n \frac{J_1^2(\alpha_n)}{2} \tag{34}$$

To evaluate this final integral, we will first split it up into to integrals and tackle them one by one. Notice that the above equation is equivalent to:

$$\int_0^1 \xi J_0(\alpha_m \xi) d\xi - \int_0^1 \xi^3 J_0(\alpha_m \xi) d\xi = B_n \frac{J_1^2(\alpha_n)}{2} \tag{35}$$

It is clear from Eq. (R3) that the first integral can be evaluated to:

$$\int_0^1 \xi J_0(\alpha_m \xi) d\xi = \frac{1}{\alpha_n} J_1(\alpha_n) \tag{36}$$

To solve the second integral, we first to integration by parts. We say $u = \xi^2$ and $dv = \xi J_0(\alpha_m \xi) d\xi$. Given this, integration by parts gives us:

$$\int_0^1 \xi^3 J_0(\alpha_m \xi) d\xi = \frac{J_1(\alpha_n)}{\alpha_n} - \int_0^1 \frac{2\xi^2}{\alpha_n} J_1(\alpha_n \xi) d\xi \tag{37}$$

Notice that once again, we are in the form of Eq. (R3). This means that we can evaluate this integral into:

$$\int_0^1 \xi^3 J_0(\alpha_m \xi) d\xi = \frac{J_1(\alpha_n)}{\alpha_n} - \frac{2J_2(\alpha_n)}{\alpha_n^2} \tag{38}$$

If we apply Eq. (R4) for $J_1(\alpha_n)$ we can see:

$$J_0(\alpha_n) + J_2(\alpha_n) = \frac{2J_1(\alpha_n)}{\alpha_n}$$

We know that $J_0(\alpha_n) = 0$ by definition, so we can reduce this to give us J_2 as a function of J_1. Given this, we can plug this in for J_2 above to get:

$$\frac{J_1(\alpha_n)}{\alpha_n} - \frac{2J_2(\alpha_n)}{\alpha_n^2} = \frac{J_1(\alpha_n)}{\alpha_n} - \frac{4J_1(\alpha_n)}{\alpha_n^3} \tag{39}$$

Now, recall that we wanted to evaluate:

$$\int_0^1 \xi J_0(\alpha_m \xi) d\xi - \int_0^1 \xi^3 J_0(\alpha_m \xi) d\xi = B_n \frac{J_1^2(\alpha_n)}{2}$$

Plugging in the values we got for each integral, we arrive at:

$$\frac{J_1(\alpha_n)}{\alpha_n} - \frac{J_1(\alpha_n)}{\alpha_n} + \frac{4J_1(\alpha_n)}{\alpha_n^3} = B_n \frac{J_1^2(\alpha_n)}{2} \tag{40}$$

From here, solving for B_n is simple and we get:

$$B_n = \frac{8}{\alpha_n^3 J_1(\alpha_n)} \tag{41}$$

Putting this all together with the steady state solution and the time-dependent terms we get:

$$\phi = (1 - \xi^2) - \sum_{n=1}^{\infty} \frac{8J_0(\alpha_n \xi)}{\alpha_n^3 J_1(\alpha_n)} e^{-\alpha_n^2 \tau} \tag{42}$$

Below, we have shown the result of plotting the normalized velocity against the radial position at various times.

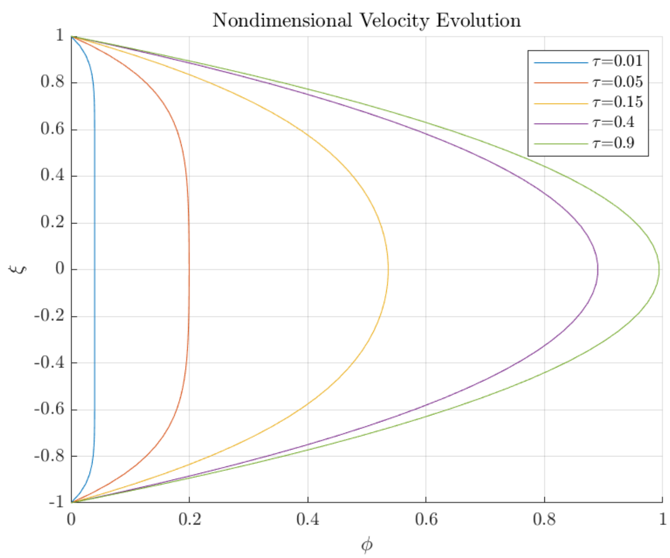

Figure 1: Flow Profile Evolution

From the above figure, we can see the expected non-dimensional velocity evolution. It is important to note that we have plotted ϕ on the x axis and
xi on the y axis. The reason we did this, when the problem asks for v/v_{max} instead is that ϕ is equivalent to this non-dimensionalization. Let us recall that V_{max} is the maximum velocity of the steady state solution. We found that

$$\phi_\infty = 1 - \xi^2$$

From this, it is clear to see that the velocity is maximized when ϕ is maximized, and that ϕ is maximized when $\xi = 0$. Given this, we can say

$$v_{max} = \frac{(P_0 - P_L)R^2}{4\mu L} \tag{43}$$

From this, it is clear to see that $v/v_{max} = \phi$. As such the above plot is what we expected to get to.

From the behavior of this graph, and the time stamps on teach curve, we can see that the time it takes to develop is fairly quick. We know we have reached steady state once we have reached $\phi = 1$ at the peak, which has just about happened at $\tau = 0.9$. However, the time it takes to get to $\phi_{max} \approx 0.9$ is about the same time from the start at it takes to go from $\phi_{max} = 0.9$ to $\phi_{max} = 1$. This is expected of the exponential decay nature of the system.

In addition to this "slowing-down" behavior, we can see that during the development region, there is a curved section near the wall and a constant section near the center. This is a result of the boundary layer growing and "including" more of the flow in the curved profile. At full development, the boundary layer has not fully developed to include the entire pipe. As such, section outside of the boundary layer can have the uniform "freestream" velocity. However, as we go on in time, the boundary layer includes the entire pipe but now the entire system is still accelerating together. It may be interesting to look if there is a relationship between the entrance length of a pipe and the time at which we would find a velocity profile with no constant core section. However, this analysis is beyond the scope of this homework.

MATLAB Code used to visualize the results of this homework is included in the Appendix.

Appendix

MATLAB Code

```
clear; close all; clc

%% Numerical Approximations
Nterms = 100;
Npoints = 50;
Ntime = 5;

%% Basis Values
P0 = 10;
PL = 0;
L = 1;
R = 1;
rho = 1;
mu = 1;

Vmax = (P0-PL)*R^2/(4*mu*L);

%% Spans
rset = linspace(-R,R,Npoints);
xiset = rset/R;
tauset = [0.01, .05, .15,.4,.9];
%% Find Solution
% Zeros of the Bessel Function
alpha = besselzero(0,Nterms,1);
timeConstant = 1/alpha(1)^2;


Bn = 8./(alpha.^3.*besselj(1,alpha));

% Steady State Terms
phiset = zeros(Npoints,Ntime); % Location, Time Step
phiinfty = 1-xiset.^2;

for i = 1:Ntime
    phit = zeros(Npoints,1);
    for j = 1:Nterms
       exponential = exp(-alpha(j)^2*tauset(i));
       bessel = besselj(0,alpha(j)*xiset)';

       phit = phit + Bn(j)*bessel*exponential;
    end

    phiset(:,i) = phiinfty'-phit;
end

%% Plot
set(0,'defaultTextInterpreter','latex');
set(0,'defaultAxesTickLabelInterpreter','latex');
set(0,'defaultLegendInterpreter','latex');
colors = turbo(Ntime);
% Plot Phi-xi
figure;
```

```
hold on;
ylabel("$\xi$");
xlabel("$\phi$");
title("Nondimensional Velocity Evolution");
for i = 1:Ntime
    plot(phiset(:,i),xiset,'DisplayName',"$\tau$="+num2str(round(tauset(i)*100)/100));
end
legend("Location","northeast");
grid on;
exportgraphics(gcf,"NondimAll.png");
% Dim Time
tConst = rho*R^2/mu;
figure;
hold on;
ylabel("$\xi$");
xlabel("$\phi$");
title("Nondimensional Velocity Evolution");
for i = 1:Ntime
    plot(phiset(:,i),xiset,'DisplayName',"t="+num2str(round(tConst*tauset(i)*100)/100)+"s");
end
legend("Location","northeast");
grid on;
exportgraphics(gcf,"DimTime.png");
% Dim All
tConst = rho*R^2/mu;
figure;
hold on;
xlabel("$V$");
ylabel("$r$");
title("Velocity Evolution");
for i = 1:Ntime
    plot(phiset(:,i)*Vmax,xiset*R,'DisplayName',"t="+num2str(round(tConst*tauset(i)*100)/100)+"s");
end
legend("Location","northeast");
grid on;
exportgraphics(gcf,"DimAll.png");
```