MEAM 5360: VISCOUS FLUID FLOW

Assignment 1: Conservation Laws

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Problem 1

Prove that, for a Newtonian and incompressible fluid, the relation $\tau_{ij} = \tau_{ji}$ is valid. Here, τ is the shear stress. What are the consequences or implications of this identity? Please show all the steps and assumptions.

To begin this problem, consider a 3D rectangular fluid element. This fluid element has side lengths dx and dy. This fluid has viscous stresses acting parallel and perpendicular to each face. However, since the fluid is Newtonian and we are ignoring pressure, we know τ_{xx} and τ_{yy} are both zero. If we take the moment about one of the Edges, we can then say:

$$\sum M = \tau_{xy} dydzdx - \tau_{yx} dxdzdy = I\dot{\omega} \tag{1}$$

In this case, we have taken the moment about an edge along the z direction. However, the equation does not change significantly depending on the edge one chooses. For a cube about its edge, the moment of inertia is given by:

$$I = m\frac{dx^2 + dy^2}{12} \tag{2}$$

Plugging this into the above equation, we get:

$$\tau_{xy} - \tau_{yx} = \frac{m}{dxdydz}\frac{dx^2 + dy^2}{12}\dot{\omega} \tag{3}$$

Notice that we can define the density of our fluid as $m/(dxdydz)$. For an incompressible fluid, this would be a constant value we will call ρ. This simplify our equation to

$$\tau_{xy} - \tau_{yx} = \rho\frac{dx^2 + dy^2}{12}\dot{\omega}$$

From here, we can apply the continuity hypothesis and take the limit as dx and dy approach zero. By substitution, it is clear that the right-hand side will reduce to zero in this limit. This means we get:

$$\tau_{xy} - \tau_{yx} = 0 \implies \tau_{xy} = \tau_{yx} \tag{4}$$

If we repeat this process for edges parallel to the x and y axes also, we will see we get $\tau_{yz} = \tau zy$ and $\tau_{xz} = \tau_{zx}$ respectively. This so far has shown that the viscous stress tensor is symmetric. In addition, since we know that the stresses caused by pressure only influence terms on the main diagonal, we know that if the viscous stress tensor is symmetric, the total stress tensor will also be symmetric.

Problem 2

Introduction

In this problem, we will be looking at the flow field and shear stress profile of a parallel plate rheometer. We will look at the torque required to operate this parallel plate geometry along with looking at the limiting characteristics of this type of flow. At the end there will be a results section, summarizing all results as asked for in the assignment in a neat format without all derivations.

To approach this problem, we start with the continuity equation, see how this simplifies our momentum equations, and then solve the simplified momentum equations to find the velocity profile. Once we have the velocity profile, we will then find the shear stress profile and analyze the profiles and the torque from here. Below, we list the assumptions we will make in this problem.

Assumptions

1. Incompressible
2. Newtonian
3. Steady-State

4. $v_r = 0$
5. $v_z = 0$
6. $\frac{d}{d\theta} = 0$

Continuity Equation

Due to the rotational symmetry, we will use the cylindrical form of the continuity equation. The same will be true for the Navier-Stokes equation once we reach that point. In cylindrical coordinates, the continuity equation is given by:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{1}$$

By Assumptions 1, 4, and 5 we know that the first term, second term, and final term go to zero. In addition, the second term can be simplified to:

$$\frac{\partial v_\theta}{\partial \theta} = 0 \tag{2}$$

This is the most simplified form of our continuity equation and gives us proof that our system is independent of the θ-wise position, as expected. Now we move on to the Navier-Stokes Equations

Navier-Stokes Equation

Now we will look at each component of our Navier-Stokes Momentum Equations. We will begin with the simpler ones in the z and r momentum, as we know the velocity in these directions. Finally, we will look at the θ direction as it will require the bulk of our analysis.

z-Momentum Equation

In full, the z-Momentum equation in cylindrical coordinates can be written as:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial \theta^2}\right] + \rho g_z \tag{3}$$

Since we know $v_z = 0$, all terms other than the pressure and gravity terms go to zero. This means, that we can write the z-momentum equation as:

$$\frac{\partial p}{\partial z} = \rho g_z \tag{4}$$

This is simply a statement of the hydrostatic pressure, resulting from the symmetry of the problem. This also shows us the pressure is linear in the z direction. However, we are not particularly concerned about the pressure in this problem as it is not a necessary component of our analysis.

r-Momentum Equation

In full, the r-Momentum equation in cylindrical coordinates can be written as:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] \tag{5}$$

Once again, we know that $v_r = 0$ so most of these terms go to zero. The centrifugal term, the pressure term, and the third term in the viscous stresses remain. This gives us:

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial p}{\partial r} - \frac{2\mu}{r^2}\frac{\partial v_\theta}{\partial \theta} \tag{6}$$

This seems like a partial differential equation in terms of the θ velocity and the pressure. However, if we recall the final form of our continuity equation, we can notice that the θ gradient of θ velocity is zero. THIs means our r-Momentum equation simplifies to:

$$\frac{\partial p}{\partial r} = \frac{\rho v_\theta^2}{r} \tag{7}$$

This has now uncoupled the radial pressure gradient from the θ velocity. Once again, since we do not need the pressure for this problem we leave this problem here.

θ-Momentum Equation

In full, the θ-Momentum equation in cylindrical coordinates can be written as:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right] \tag{8}$$

Since this is the primary equation we will be looking at, we will go term by term to see what cancels out. On the left, the first term is zero by assumption 3. The second term is zero by assumption 4. The third term is zero by Eq. (2). The fourth term is zero by assumption 4. Finally, the fifth term is zero by assumption 5. This shows that the left-hand side is zero. We will now look at the right-hand side. The pressure gradient is zero due to the axisymmetric nature of the problem captured in assumption 6. Inside the viscous stress brackets, only terms that are zero is the second term, following from Eq. (2) as the derivative of zero is zero, and the third term, by assumption 4 Since all other terms are zero, we can also cancel out the μ term. This gives us the Partial Differential Equation below:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{\partial^2 v_\theta}{\partial z^2}=0 \tag{9}$$

To solve this problem, we must use the separation of variables technique for PDEs. To do this, we assume our solution takes the following form:

$$v_\theta(r,z)=f(r)g(z) \tag{10}$$

This tells us that v_θ is a product of two single variable functions. If we plug this into our differential equation, we can see this gives us:

$$g\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rf)\right)+f\frac{\partial^2 g}{\partial z^2}=0 \tag{11}$$

Through rearrangement, we get this into the form:

$$\frac{1}{f}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rf)\right)=-\frac{1}{g}\frac{\partial^2 g}{\partial z^2} \tag{12}$$

Since the left-side is purely a function of r and the right-side is purely a function of z, we know that each these values must be equal to some constant. This is the basis of PDE Eigenvalue problems. We say that both sides of this equation will be equal to some value λ^2. Given this, we can separate these equations into the following system:

$$\frac{1}{f}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rf)\right)=\lambda^2 \tag{13}$$

$$-\frac{1}{g}\frac{\partial^2 g}{\partial z^2}=\lambda^2 \tag{14}$$

These equations are now ODEs that can be solved independently. However, they are coupled through the value of λ. Since this is a constant, it does not introduce any complex non-linearity, but it is something important to notice.

Before we go on to solving these equations, we will first analyze the boundary conditions. To make boundary conditions easier, instead of putting the origin centered on the bottom plate, which rotates, we placed the origin at the top plate, which is fixed. This allowed us to say $v_\theta(r,0)=v_\theta(R,z)=0$ as our two stationary surfaces. We also require the solution to stay bounded at $r=0$. Finally, for the bottom face, we say $v_\theta(r,-H)=r\omega$. This gives us all the boundary conditions we require. For our simplified equations, we can get several boundary conditions. First, we require that $f(0)$ be bounded. Second, we require $f(R)=0$ for the wall boundary condition to be met. Finally, we can say $g(0)=0$. We cannot enforce bottom boundary condition since it is dependent on both a z value and an r value, requiring a coupled solution to enforce. Now that we have boundary conditions, we can go about solving these problems. First, let us solve the g equation.

Rearranging Eq. (14), we get:

$$\frac{d^2 g}{dz^2}+\lambda^2 g=0 \tag{15}$$

This is a well known differential equation with the orthogonal solution function being sine and cosine. The frequency is given by λ. This gives us:

$$g=c_1\sin(\lambda z)+c_2\cos(\lambda z) \tag{16}$$

We know that $g(0) = 0$. It is clear from this that $c_2 = 0$. This means we can simplify our equation to:

$$g = c_1 \sin(\lambda z) \tag{17}$$

Since we do not have a second boundary condition for g, we cannot solve for any of the other constants. Now, let us go to the f equation. Beginning with Eq. (13) and rearranging, we can see the below result. It is important to note that, for ease, we have replaced $\partial f/\partial r$ with f'.

$$\begin{aligned}
\frac{1}{f}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rf)\right) &= \lambda^2 \\
\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rf)\right) - \lambda^2 f &= 0 \\
\frac{\partial}{\partial r}\left(\frac{f}{r} + f'\right) - \lambda^2 f &= 0 \\
\frac{rf' - f}{r^2} + f'' - \lambda^2 f &= 0 \\
r^2 f'' + rf' - \left(1 + \lambda^2 r^2\right) f &= 0 \\
r^2 f'' + rf' + \left((i\lambda r)^2 - 1^2\right) f &= 0
\end{aligned} \tag{18}$$

We put it specifically in this form as this is the form of the Bessel Differential equation. This as solution functions of the bessel functions of the first and second kinds. This gives us a solution in the form:

$$f = c_1 J_1(i\lambda r) + c_2 Y_1(-i\lambda r) \tag{19}$$

It is known that, $Y_1(0)$ is unbounded. However, we require that our solution remains bounded at $r = 0$. This means it is required that $c_2 = 0$. Applying this and also saying $f(R) = 0$ we get:

$$f(R) = c_1 J_1(i\lambda R) = 0 \tag{20}$$

If we say $c_1 = 0$, we would have $f(r) = 0$ which is a non-physical solution to our system. This means that we must instead require:

$$J_1(i\lambda R) = 0 \tag{21}$$

This means that we pick λ such that, when we evaluate f at R, the Bessel function of the first kind is zero. However, J_1 has infinite non-trivial zeros. We will define where there happens by ψ_n. This is to say ψ_n is the nth zero of J_1. Given this, we can then say

$$\begin{aligned}
i\lambda_n R &= \psi_n \\
\lambda_n &= \frac{\psi_n}{iR}
\end{aligned} \tag{22}$$

The zeros of the Bessel functions are tabulated or can be found numerically through various means. For our sakes, we use established MATLAB functions to evaluate this. Now that we have a value for λ, we can plug this into our solutions for f and g. However, it is important to note that the constant coefficients c_1 in each equation can vary with n. As such, we will add the n subscript to each constant. For simplicity, we will leave our formulas in terms of λ, but the above formula for λ_n is implied as used. To recap, we arrived at

$$f_n = c_{1,n} J_1\left(\frac{\psi_n r}{R}\right) \tag{23}$$

$$g_n = c_{1,n} \sin(\lambda_n z) \tag{24}$$

All functions f_n are orthogonal and all g_n are orthogonal. We can multiply these values together, and say that the product of the two coefficients combines into one coefficient c_n, to get the following relationship for v_θ. It is important to note that

$$v_\theta = \sum_{n=1}^{\infty} c_n J_1\left(\frac{\psi_n r}{R}\right) \sin(\lambda_n z) \tag{25}$$

We have one free variable here, but still have one boundary condition to enforce. Recall that we said $v_\theta(r, -H) = r\omega$ in our modified coordinate system. This tells us:

$$\sum_{n=1}^{\infty} c_n J_1\left(\frac{\psi_n r}{R}\right)\sin(-\lambda_n H) = r\omega \tag{26}$$

This is a Fourier-Bessel series. Since all of the Bessel functions, they create an set of functions that span all of function space. This means any function, such as $r\omega$, can be represented by some linear combination of these Bessel functions. The derivation of this will not be shown, but it can be shown that, in this case,

$$c_n \sin(-\lambda_n H) = \frac{2\int_0^R \omega r^2 J_1\left(\frac{\psi_n r}{R}\right) dr}{\left(RJ_2(\psi_n)\right)^2} \tag{27}$$

$$c_n = \frac{2\int_0^R \omega r^2 J_1\left(\frac{\psi_n r}{R}\right) dr}{\sin(-\lambda_n H)\left(RJ_2(\psi_n)\right)^2} \tag{28}$$

Although not analytical, this has given us full definition of our system. This can be solved numerically using MATLAB to evaluate Bessel functions, integrals, find zeros of the Bessel functions, and sum the many terms. To recap, we found:

$$v_\theta(r, z) = \sum_{n=1}^{\infty} c_n J_1\left(\frac{\psi_n r}{R}\right)\sin(\lambda_n z) \tag{29}$$

$$c_n = \frac{2\int_0^R \omega r^2 J_1\left(\frac{\psi_n r}{R}\right) dr}{\sin(-\lambda_n H)\left(RJ_2(\psi_n)\right)^2} \tag{30}$$

$$\lambda_n = \frac{\psi_n}{iR} \tag{31}$$

$$J_1(\psi_n) = 0 \tag{32}$$

With this being fully defined, we have completed our analysis of the θ-Momentum equation and found an expression for v_θ. The next step is to find the shear stress profiles. In those steps, we will take this equation for granted.

Shear Stress

To begin, recall that in most of the components of the Navier-Stokes equations, the viscous stress terms went to zero. The remaining terms correspond to the $\tau_{r\theta}$ and $\tau_{z\theta}$ components. These are the only two non-zero shear stress values. In cylindrical coordinates, given our assumptions, these can be expressed as:

$$\tau_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) \tag{33}$$

$$\tau_{z\theta} = \mu \frac{\partial v_\theta}{\partial z} \tag{34}$$

We will first do $\tau_{z\theta}$, as it only requires a simple derivative. Taking this derivative, we get:

$$\tau_{z\theta} = \mu \sum_{n=1}^{\infty} c_n \lambda_n J_1\left(\frac{\psi_n r}{R}\right)\cos(\lambda_n z) \tag{35}$$

Finding the other shear stress is more involved as it requires product rule and the derivative of the Bessel function. To do this, let us break this up first. We can say:

$$\tau_{r\theta} = \mu r \left(\frac{\frac{\partial v_\theta}{\partial r} r - v_\theta}{r^2}\right) = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) \tag{36}$$

Next, we can find the derivative of the Bessel function by looking on the internet. This gives us:

$$\frac{d}{dr} J_1\left(\frac{\psi_n r}{R}\right) = \frac{\psi_n}{2R}\left(J_0\left(\frac{\psi_n r}{R}\right) - J_2\left(\frac{\psi_n r}{R}\right)\right) \tag{37}$$

From this, we can find the velocity gradient as:

$$\frac{\partial v_\theta}{\partial r} = \sum_{n=1}^{\infty} \frac{c_n \psi_n}{2R} \left(J_0\left(\frac{\psi_n r}{R}\right) - J_2\left(\frac{\psi_n r}{R}\right)\right) \sin(\lambda_n z) \tag{38}$$

We can then subtract the second term and arrive at a shear stress:

$$\tau_{r\theta} = \mu \sum_{n=1}^{\infty} \frac{c_n \psi_n}{2R} \left(J_0\left(\frac{\psi_n r}{R}\right) - J_2\left(\frac{\psi_n r}{R}\right)\right) \sin(\lambda_n z) - \mu \sum_{n=1}^{\infty} \frac{c_n}{r} J_1\left(\frac{\psi_n r}{R}\right) \sin(\lambda_n z) \tag{39}$$

Putting all of this under one sum, we arrive at:

$$\tau_{r\theta} = \mu \sum_{n=1}^{\infty} \frac{c_n \psi_n}{2R} \left(J_0\left(\frac{\psi_n r}{R}\right) - \frac{2R}{\psi_n r} J_1\left(\frac{\psi_n r}{R}\right) - J_2\left(\frac{\psi_n r}{R}\right) \sin(\lambda_n z)\right) \tag{40}$$

Torque

The final step is to find the Torque required to operate such a device. We know that the shear stress that provides the resistance in this case is $\tau_{z\theta}$ as we are driving the bottom face, which has a z normal vector. From this, we then want to integrate this component of the shear stress over the entire bottom face. However, this will give us the shear stress the fluid exerts on the bottom plate. Since we want to the torque required to drive this derive, we must take the negative of this value. In equations, this becomes:

$$T = -\iint_{bot} \tau_{z\theta} dA \tag{41}$$

Plugging in the proper bounds and equations, this becomes:

$$T = -\int_0^R \int_0^{2\pi} \mu \sum_{n=1}^{\infty} c_n \lambda_n J_1\left(\frac{\psi_n r}{R}\right) \cos(-\lambda_n H)\, d\theta dr \tag{42}$$

Since our function is independent of θ, we can evaluate the θ integral. In addition, we will apply the identity that $\cos(x) = \cos(-x)$. Together, this puts us at:

$$T = -2\pi \int_0^R \mu \sum_{n=1}^{\infty} c_n \lambda_n J_1\left(\frac{\psi_n r}{R}\right) \cos(\lambda_n H)\, dr \tag{43}$$

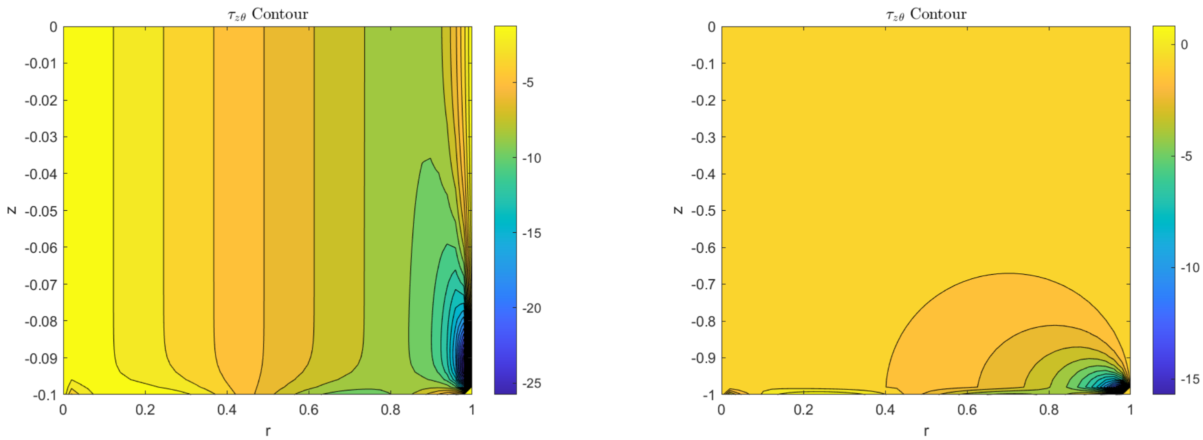
In the case of a Newtonian Fluid, we can take the viscosity out of the integral. This means that the integral can be evaluated a-priori depending on the geometric specifications of the geometer. From there, we can measure the and divide by the geometrical constant of the rheometer to arrive at the viscosity.

Results

Here, we will collect the results from the above analysis into a neat format. This is also where we will include images for the velocity and shear stress profiles. Equations will be taken as needed from the above sections. For the following results, we will be looking at a Newtonian fluid with $\mu = 1$. We will take the Rheometor radius fixed at $R = 1$ but take two values for H; one where $H = 0.1$ and one where $H = 1$. We will compare the vertical velocity profiles between our exact case and the results found by making a thin film approximation to see where it would be appropriate. This will also be seen in the shear stress profiles. In addition, to numerically calculate the values we took a 50x50 grid in both the r and z directions to get a good contour for the results. For the infinite series, we took the first 100 terms. Even at this points, the values of c_n were approaching 10^{-75}. This means we have taken more than enough points, and could likely get away with less but the computational cost was negligible. We will now look at the resulting graphs.

a. Stress Distribution

By evaluating our closed-form function for the shear stresses at all points of interest, we can create a contour plot of the stress distribution. Results for $\tau_{z\theta}$ can be found in Figure 1. As can be seen, increasing the Aspect Ratio leads

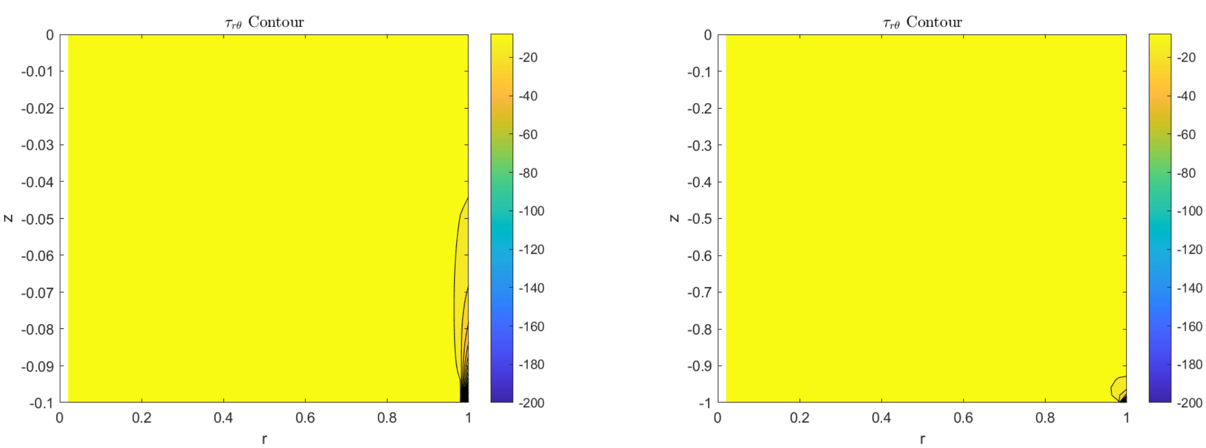


(a) $H/R = 0.1$ **(b)** $H/R = 1$

Figure 1: $\tau_{z\theta}$ for Varying Aspect Ratios

to the shear stress being much more local to the bottom plate. It is interesting to note that, in the thin case, there are significant gradients in the shear stress near the wall at R. This is a result of the no-slip boundary condition at that wall interacting oddly with the imposed velocity of ωR at the bottom plate, leading to large velocity gradients in the r direction that also vary in the z direction.

A similar analysis behavior can be seen in $\tau_{r\theta}$, shown in Figure 2. A major difference between the $\tau_{r\theta}$ and $\tau_{z\theta}$ components is both in the range of locations and range of values taken on. $\tau_{r\theta}$ has much larger values, but is localized to the intersection of the velocity discontinuity. This is understandable as $\partial v_\theta/\partial r$ is very large as we approach $(r, z) = (R, -H)$. However, the value is much more tame away from the walls. In fact, the values away from the discontinuity are roughly constant, indicating the roughly constant velocity gradient we expect. In terms of the equations that describe these fields, we can refer to Eq. (35) and Eq. (40) along with Eq. (29),Eq. (30),Eq. (31),Eq. (32) to get the entire picture.

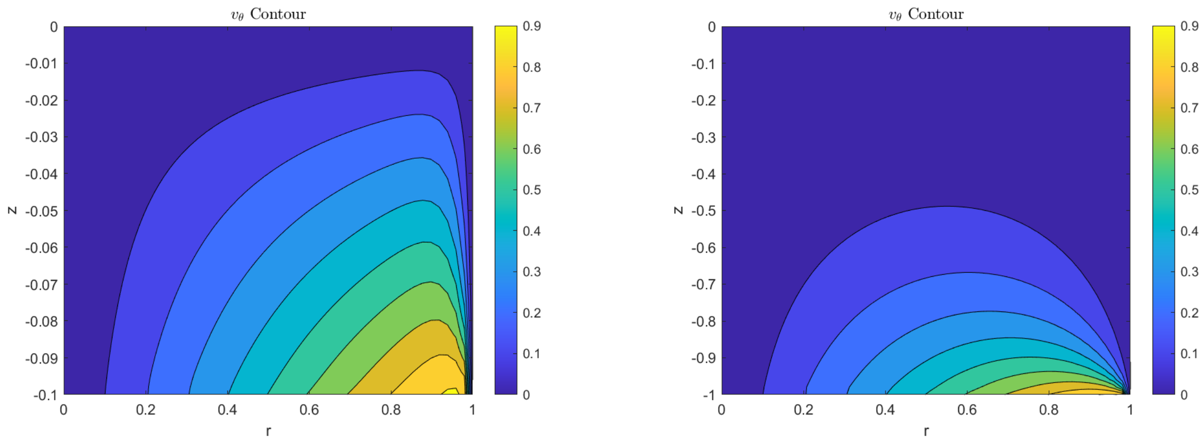


(a) $H/R = 0.1$ **(b)** $H/R = 1$

Figure 2: $\tau_{r\theta}$ for Varying Aspect Ratios

b. Velocity Distribution

The shear stress provided us interesting insights into the behavior of the flow, but more insight can be gained by looking at the velocity field. The velocity contour between the plates can be seen in Figure 3. As can be seen, increasing the aspect ratio leads to the rotational perturbations staying closer to the wall. This is expected, a viscosity plays the roll of diminishing the influence further away from the wall. Although the contours give us a decent idea of how the flow behaves as we vary the Aspect Ratio, more insight can be gained from looking at the



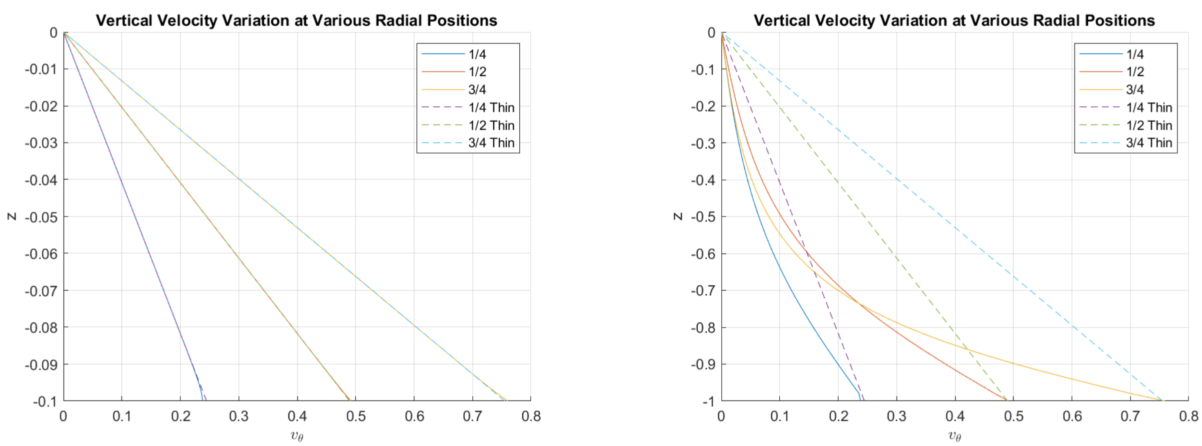
(a) $H/R = 0.1$ **(b)** $H/R = 1$

Figure 3: v_θ for Varying Aspect Ratios

flow at various radial positions. In addition, we will compare this to the simpler, analytical solution that is found by making the thin-film assumption for this geometry. This gives us:

$$v_\theta = -r\omega \frac{z}{H} \tag{44}$$

This is also in our same coordinate system. It is clear that this is a much simpler equation, but the drawback is that the second radial boundary condition, the no-slip at R is not satisfied. However, when $R >> H$, this error is often negligible. When we evaluate the velocity we can then evaluate the velocity profiles at $R/4, R/2$, and $3R/4$. This gives us the results in Figure 4.



(a) $H/R = 0.1$ **(b)** $H/R = 1$

Figure 4: v_θ at Various Radial Positions

From this figure, it is clear to see that for smaller aspect ratio, the thin approximation works well. However, for larger aspect ratios, the thin film approximation is very bad in comparison to the exact solution in two dimensions. In addition to this, recall that the thin approximation did not enforce the no-slip condition radially. Looking at the velocity along the bottom boundary, we get the result in Figure 5. From the graph, we can see the linear variation with radial position. However, very close to the wall, at approximately r=0.95, we see the velocity drop quickly to zero. If one adds more terms to the infinite series, this behavior gets closer and closer to the wall, giving an infinite slope once we reach an infinite number of terms. Of course, this would not be physical, but the discontinuity in our boundary conditions is also not perfectly physical. Dealing with this discontinuity is outside the scope of this problem, but an interesting potential idea.

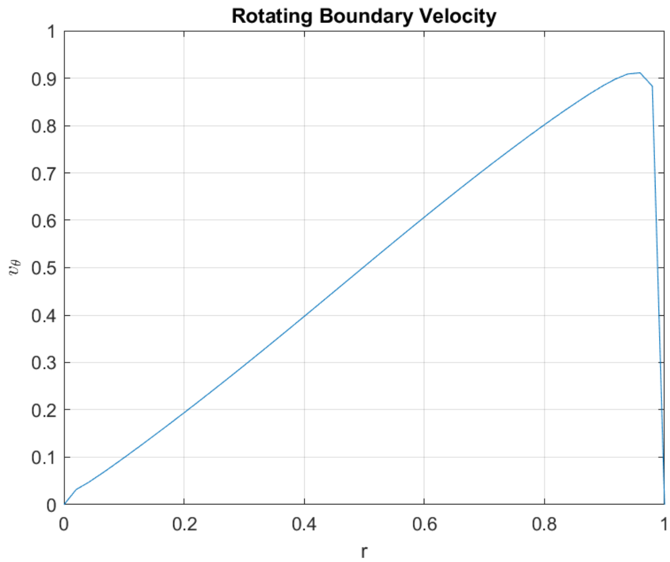


Figure 5: Boundary Velocity at Bottom Plate

To connect this graph to what we saw in our shear stress diagrams, recall that we found very large values for $\tau_{r\theta}$. This is primarily a result of this large gradient at the wall, but a constant gradient elsewhere. This is why we see a constant shear stress everywhere except for near the radial wall, where the shear stress, and by inference the velocity gradient is large.

c. Torque

Recall that we derived a closed-form relationship for the torque through Eq. (43) which is recalled here:

$$T = -2\pi \int_0^R \mu \sum_{n=1}^{\infty} c_n \lambda_n J_1\left(\frac{\psi_n r}{R}\right)\cos(\lambda_n H)\, dr$$

For a Newtonian fluid, this the viscosity can be removed from the integral and the formula can be written as:

$$T = -\mu \left[2\pi \int_0^R \sum_{n=1}^{\infty} c_n \lambda_n J_1\left(\frac{\psi_n r}{R}\right)\cos(\lambda_n H)\, dr\right] \tag{45}$$

The term in brackets is a constant value dependent only on the geometry of the Rheometer. As such, it can be calculated very accurately once and the value saved for future use. This means this geometry would be very easy to use for Newtonian fluids.

Blood?

Blood is often described as a visco-elastic fluid, as there are polymers in the solution that have elastic properties that influence the flow and can also have preferred directions, resulting in anisotropic behavior of the stress tensor. If one was to try to fit something like a power law or have an effective viscosity value that they would use, they would need to have it as a function of shear rate. That is to say: $\mu = \mu(\dot{\gamma})$. Given that we also know that the shear rate is significantly varying radially, especially near the edges, we can no longer take the viscosity out of the integral in the Torque equation. This means that we can no longer find a "Geometric Constant" for the Rheometer, as the viscosity is no longer separate.

To put some intuition onto this, we know from past experiences that red blood cells, which we can treat like particles in a homogeneous blood plasma, would be pushed toward the outer edges of the rheometer as it has a larger mass than the water around it, meaning pressure is not enough to keep it still. As red blood cells build up, they would rub against each other or cause small gaps that the flow would have to pass through, increasing the stress in that local area. This means that the effective viscosity would be much larger at the edges than would be larger, leading to a radial viscosity difference which we need to account for in our integral. Overall, this makes **our geometry a bad geometry to measure the viscosity of blood.**

Appendix

Problem 2 MATLAB

```
clear; close all; clc;

N =50;
nterms = 100;
R=1;
H=.1;
omega=1;
mu = 1;

rmin = 0;
rmax = R;
zmin = -H;
zmax = 0;
xs = linspace(rmin,rmax,N);
ys = linspace(zmin,zmax,N);
xset = zeros(N);
yset = zeros(N);

for i = 1:N
    for j = 1:N
        xset(i,j) = xs(i);
        yset(i,j) = ys(j);
    end
end

% First nterms Zeros of the Bessel Functions. We truncate after this.
psi = besselzero(1,nterms,1);

% Eigenvalues of associated terms above.
lambda = psi/(1i*R);

% Fourier-Bessel Coefficients of the associated terms
fun = @(x,n) (omega.*x.^2.*besselj(1,psi(n).*x/R));

C = zeros(nterms,1);
for i = 1:nterms
    numer = integral(@(x) fun(x,i),0,R);
    denom = 0.5*sin(-lambda(i).*H).*(R.*besselj(2,psi(i))).^2;
    C(i) = numer/denom;
end

vtheta = zeros(N);
vthetaThin = zeros(N);
tau_zt = zeros(N);
tau_rt = zeros(N);
for j = 1:N
    for i = 1:N
        l = (j-1)*N+i;
        r = xset(i,j);
        z = yset(i,j);
        for k = 1:nterms
            vtheta(i,j) = vtheta(i,j) + C(k)*sin(lambda(k)*z)*besselj(1,psi(k)*
```

```
                ↪ r/R);
            tau_zt(i,j) = tau_zt(i,j) + mu*lambda(k)*C(k)*cos(lambda(k)*z)*
                ↪ besselj(1,psi(k)*r/R);
            tau_rt(i,j) = tau_rt(i,j) + mu*C(k)*psi(k)*sin(lambda(k)*z)*(
                ↪ besselj(0,psi(k)*r/R)-2*R*besselj(1,psi(k)*r/R)/(psi(k)*r)-
                ↪ besselj(2,psi(k)*r/R))/(2*R);
        end
        vthetaThin(i,j) = -r*(omega*z/H);

    end
    if(mod(l,10000) == 0 )
        disp(num2str(l)+"/"+num2str(N^2));
    end
end
% Calulate Torque
torque = 0;
stress = @(x,n) (lambda(n)*C(n).*cos(-lambda(n).*H).*besselj(1,psi(n).*x./R));
% figure;
% hold on;
for i = 1:nterms
    torque = torque + 2*pi*mu*integral(@(x) x.*stress(x,i),0,R);
    % plot(i,torque,'b.');
    % disp(torque);
end
torqueThin = (-2*pi*mu*omega*R^3)/(3*H);
%
% figure;
% hold on;
% plot(ys,(tau_zt(10,:)));
% plot(ys,(tau_zt(20,:)));
% plot(ys,(tau_zt(30,:)));
% plot(ys,(tau_zt(40,:)));

%% Figures

figure;
contourf(xset,yset,vtheta);
xlabel("r")
ylabel("z")
title('$$v_\theta$$ Contour','interpreter','latex')
colorbar;
exportgraphics(gcf,"Homework1Image_vtheta_10H="+num2str(10*H)+".png");

figure;
contourf(xset,yset,((tau_zt)),20);
xlabel("r")
ylabel("z")
title('$$\tau_{z\theta}$$ Contour','interpreter','latex')
colorbar;
exportgraphics(gcf,"Homework1Image_tau-zt_10H="+num2str(10*H)+".png");

figure;
contourf(xset,yset,(((tau_rt))),20);
xlabel("r")
ylabel("z")
title('$$\tau_{r\theta}$$ Contour','interpreter','latex')
```

```
colorbar;
exportgraphics(gcf,"Homework1Image_tau-rt_10H="+num2str(10*H)+".png");

figure;
plot(xs,real(vtheta(:,1)));
grid on;
xlabel("r");
ylabel("$$v_\theta$$",'interpreter','latex')
title("Rotating Boundary Velocity");
exportgraphics(gcf,"Homework1Image_BoundaryVelocity_10H="+num2str(10*H)+".png");

figure;
hold on;
plot((vtheta(round(N/4),:)),ys,'displayName',"1/4");
plot((vtheta(round(N/2),:)),ys,'displayName',"1/2");
plot((vtheta(round(3*N/4),:)),ys,'displayName',"3/4");
plot((vthetaThin(round(N/4),:)),ys,'--','displayName',"1/4 Thin");
plot((vthetaThin(round(N/2),:)),ys,'--','displayName',"1/2 Thin");
plot((vthetaThin(round(3*N/4),:)),ys,'--','displayName',"3/4 Thin");
xlabel("$$v_\theta$$",'Interpreter','latex');
ylabel("z")
title("Vertical Velocity Variation at Various Radial Positions");
grid on;
legend("Location",'northeast');
exportgraphics(gcf,"Homework1Image_VertVariationComparison_10H="+num2str(10*H)+".png");
```