MEAM 5360: VISCOUS FLUID FLOW

Assignment 0: Review

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Problem

This problem asks us to analyze the flow through a pipe with the center portion blocked off by an internal rod. We will consider a pipe of radius R and a blockage of radius κR. The flow is going upwards against gravity. In addition, we are considering a flow that is, in the most general case, Non-Newtonian. More precisely, we will be looking at a power law fluid with constitutive law given by (1).

$$\tau_{ij} = m\left(\dot{\gamma}\right)^n \tag{1}$$

We will look at various flow properties when $n = 1$ and when $n = 1.45$. When $n = 1$, we have a Newtonian fluid. When we have $n = 1.45$ we have a Non-Newtonian fluid that is shear-thickening since $n > 1$. If $n < 1$ we would have a shear-thinning fluid.

To approach this problem, we will begin with the continuity equation. From there, we will look to understand which stress components are nonzero in addition to making some fundamental assumptions about our flow behavior to ensure it is fully one-dimensional. We will apply this to the Cauchy Momentum equations to derive our governing equations for the flow and find the shear stress profile. Next, we will apply our constitutive law to get the velocity profile, and then do the proper processing to find any other information we need. Below a list of necessary assumptions is included for future reference

Assumptions

1. Incompressible
2. Steady-State
3. $v_r = 0$
4. $v_\theta = 0$
5. $\frac{d}{d\theta} = 0$

Continuity Equation

To begin, we start with the continuity equation. In cylindrical coordinates, this is given by Eq. (2)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{2}$$

Since we have assumed $v_r = v_\theta = 0$ and we have a steady, incompressible flow, we know that the first three terms go to zero. In addition, our incompressible assumption allows us to remove the density term from the derivatives. This simplifies our above equation to:

$$\frac{\partial v_z}{\partial z} = 0 \tag{3}$$

This is the most simplified form of our continuity equation and something we will need to take advantage of in future steps.

Cauchy Momentum Equation

The next step to this problem is looking at the Cauchy Momentum Equations. This allows us to do a first analysis without considering our constitutive law yet, simplifying the initial analysis. These equations are comprised of three components, and each one will require its own treatment. Before we do this, however, we will look at which components of shear stress we need to consider. We know from class that the strain-rate is defined in cylindrical coordinates independent of the constitutive law. Given this, we can write the components of our Shear Stress Tensor as shown in Eq. (4). This form recognizes that the stress matrix will be symmetric.

$$\bar{\tau} = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{zr} \\ \tau_{r\theta} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{\theta z} & \tau_{zz} \end{bmatrix} = m \begin{bmatrix} \left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla\cdot\vec{v})\right]^n & \left[r\frac{\partial}{\partial r}(v_\theta/r) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^n & \left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]^n \\ & \left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \frac{2}{3}(\nabla\cdot\vec{v})\right]^n & \left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]^n \\ & & \left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla\cdot\vec{v})\right]^n \end{bmatrix} \tag{4}$$

Blank cells are implied by symmetry. Based on Assumptions 3 and 4 and the continuity equation, we see this can reduce to Eq. (5)

$$\bar{\tau} = m \begin{bmatrix} 0 & 0 & \left(\frac{\partial v_z}{\partial r}\right)^n \\ & 0 & 0 \\ & & 0 \end{bmatrix} \tag{5}$$

Now that we know the majority of our stress components are zero, our Cauchy Momentum Equations are simplified significantly. We will now look at the three different components of the momentum equations and see what each one tells us about the flow.

One final detail about the behavior of stress can be revealed by taking the derivative in the z Direction. This gives us the following relationship:

$$\frac{\partial \tau_{zr}}{\partial z} = \frac{\partial}{\partial z}\left(m\left(\frac{\partial v_z}{\partial r}\right)^n\right) = mn\left(\frac{\partial v_z}{\partial r}\right) \cdot \frac{\partial}{\partial r}\left(\frac{\partial v_z}{\partial z}\right) = 0 \tag{6}$$

From Eq. (3), we know that this will go to zero. This information will be important in the following section.

r-Momentum Equation

The r- Momentum Equation in Cylindrical Coordinates is given by:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \left(\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{zr}}{\partial z}\right) + \rho g_r \tag{7}$$

From Assumptions 3 and 4 we know that the entire left-hand side will be zero. We cannot make any claims about the pressure yet, but we can say that $\rho g_r = 0$ and that all the derivatives of stress are zero either due to Eq. (5) or Eq. (6). Given this, our r momentum equation reduces to Eq. (8).

$$\frac{\partial p}{\partial r} = 0 \tag{8}$$

θ-Momentum Equation

The θ-Momentum Equation in Cylindrical Coordinates is given by:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{\partial p}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right) + \rho g_r \tag{9}$$

From Assumptions 3 and 4 we know that the entire left-hand side will be zero. We cannot make any claims about the pressure yet, but we can say that $\rho g_\theta = 0$ and that all the derivatives of stress are zero due to Eq. (5). This reduces our equation to Eq. (10).

$$\frac{\partial p}{\partial \theta} = 0 \tag{10}$$

This fact, combined with our conclusion from the previous section tells us that p is only a function of z. This will be important when deriving our velocity profile.

z-Momentum Equation

The z-Momentum Equation in Cylindrical Coordinates is given by:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left(\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \tag{11}$$

Due to Assumptions 2,3, 4, and Eq. (3) we know that the left-hand size reduces to zero. Once again, the pressure remains untouched, however it is important to note that we have established this pressure gradient is only a function of z. The τ_{rz} term says also, while the other two stress terms go away. In addition, we have to keep the gravity term. This gives us the simplified form of:

$$\frac{\partial p}{\partial z} - \rho g_z = \frac{1}{r}\frac{\partial (r\tau_{zr})}{\partial r} \tag{12}$$

From here, we can solve for the shear stress profile as follows:

$$\frac{\partial(r\tau_{zr})}{\partial r} = r\left(\frac{\partial p}{\partial z} - \rho g_z\right)$$

$$r\tau_{zr} = \frac{r^2}{2}\left(\frac{\partial p}{\partial z} - \rho g_z\right) + C_1$$

$$\tau_{zr} = \frac{r}{2}\left(\frac{\partial p}{\partial z} - \rho g_z\right) + \frac{C_1}{r} \tag{13}$$

This shows us that our shear stress profile is independent of our constitutive law. However, the constitutive law will give us the velocity profiles. To do this, recall Eq. (1) and Eq. (5). From this, we can say:

$$m\left(\frac{\partial v_z}{\partial r}\right)^n = \frac{r}{2}\left(\frac{\partial p}{\partial z} - \rho g_z\right) + \frac{C_1}{r} \tag{14}$$

With some rearrangement, this can be solved for the velocity gradient as:

$$\frac{\partial v_z}{\partial r} = \left(\frac{r}{2m}\left(\frac{\partial p}{\partial z} - \rho g_z\right) + \frac{C_1}{mr}\right)^{1/n} \tag{15}$$

Since we cannot claim that $C_1 = 0$, and we will show for the Non-Newtonian case this is actually not true, we have to find the integral of this entire term with no simplifications. If one puts this integral into Wolfram Alpha, the result is a complicated function including a non-elementary function known as the Hypergeometric Function. Due to a lack of familiarity with the properties of this function, we can essentially claim this is an unsolvable problem in the general case and would require numerical solutions in some capacity to solve. This means that the Non-Newtonian case is also not solvable with the information assumed in this class, in general. To demonstrate we understand the process of solving this problem, we have still completed it for the Newtonian case as this provides a necessary simplification that enables the discovery of an analytical solution.

Newtonian

In the Newtonian case, we have n=1 which simplifies our differential equation Eq. (15). From the Cauchy-Momentum Equations, we are able to derive the stress distribution of the fluid along the pipe in the rz direction as can be seen from Eq. (13). Using Eq. (8) and Eq. (10), because there is no pressure change in the r and θ direction, we can deduce that pressure is only a function of z; $P = P(z)$ and we can introduce a new variable P_2 which is a sum of the pressure terms in Eq. (15) i.e $P_2 = \frac{\partial p}{\partial Z} - \rho g_z$

$$\frac{\partial v_z}{\partial r} = \frac{rP_2}{2m} + \frac{C_1}{mr} \tag{16}$$

The is a simple first order linear differential equation that can be solved analytically.

$$v_z(r) = \frac{r^2 P_2}{4m} + \frac{C_1}{m}\ln(r) + C_2 \tag{17}$$

In order to find the value of C_1 and C_2, two boundary conditions will be applied for the velocity at different radial locations by use of the no-slip conditions at the walls of the pipe.

$$v(\kappa R) = 0$$
$$v(R) = 0$$

We can then plug this into Eq. (17) giving us two simultaneous equations as shown below

$$0 = \frac{R^2 P_2}{4m} + \frac{C_1}{m}\ln(R) + C_2$$
$$0 = \frac{(\kappa R)^2 P_2}{4m} + \frac{C_1}{m}\ln(R) + C_2$$

Solving this, we get the values of C_1 and C_2

$$C_1 = \frac{R^2 P_2 (1 - \kappa^2)}{4 \ln(k)} \tag{18}$$

$$C_2 = -\frac{R^2 P_2}{4m} \left(1 + \frac{(1 - \kappa^2) \ln(R)}{\ln(\kappa)} \right) \tag{19}$$

These equation values can be used in Eq. (17) to find the exact velocity profile. For the sake of saving space, we have not included the full equation here but will be included in some future steps.

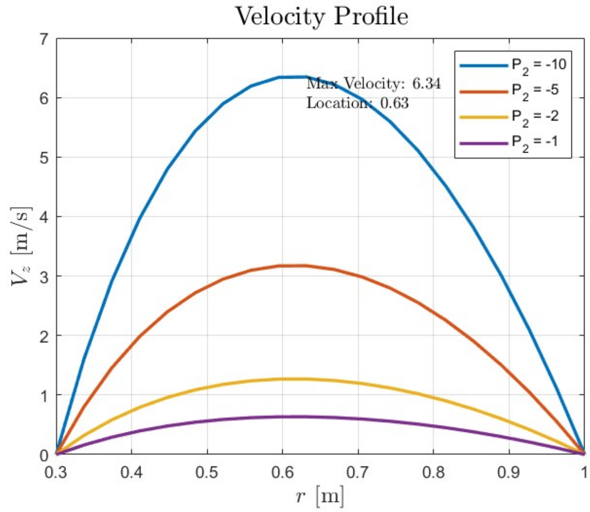

Figure 1: Velocity Profile at different P_2 values given $R = 1$, $m = 0.1$ and $\kappa = 0.3$

We can use Eq. (12) to obtain an equation for the stress distribution. This first order ODE can be solved for τ_{rz}

$$P_2 = \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) \tag{20}$$

$$\tau_{rz} = \frac{r P_2}{2} + \frac{C_1}{r} \tag{21}$$

We already solved for the value of C_1 as described in Eq. (18).

$$\tau_{rz} = P_2 \left(\frac{r}{2} + \frac{R^2 (1 - \kappa^2)}{4r \ln(k)} \right) \tag{22}$$

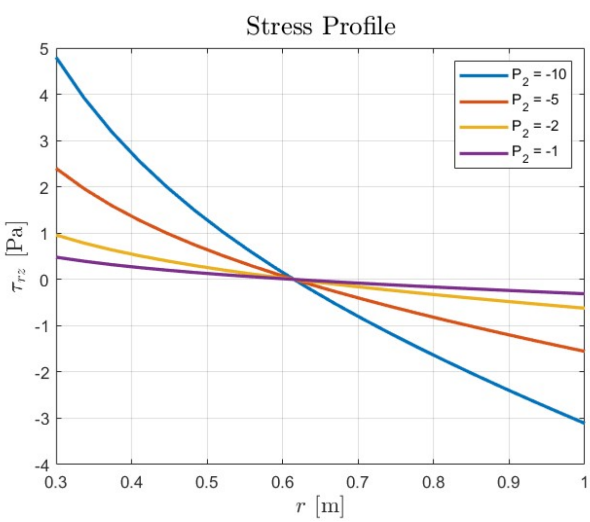

Figure 2: Stress Profile at different P_2 values given $R = 1$, $m = 0.1$ and $\kappa = 0.3$

To find the maximum velocity from here, we can use the fact that the maximum occurs when the velocity gradient is zero. This also means that this occurs when the shear stress is zero. Given this, we can set $\tau_{rz} = 0$ and find the value of r which makes this true:

$$0 = P_2\left(\frac{r}{2} + \frac{R^2(1-\kappa^2)}{4r\ln(k)}\right)$$

$$0 = \frac{r^2}{2} + \frac{R^2(1-\kappa^2)}{4\ln(k)}$$

$$-\frac{R^2(1-\kappa^2)}{2\ln(k)} = r^2$$

$$R_{max} = R\sqrt{\frac{\kappa^2-1}{2\ln(\kappa)}} = \lambda R \tag{23}$$

To find the exact value for the maximum velocity, one would simply plug R_{max} into Eq. (17) while using Eq. (18) and Eq. (19) where necessary. One should let a computer do this math.

To find the average velocity, we simply need to integrate the velocity profile over the cross-sectional area, and then divide by the total area. This gives us the relationship:

$$\bar{v}_z = \frac{1}{\pi(R^2-(\kappa R)^2)}\int_0^{2\pi}\int_{\kappa R}^{R}\left(\frac{r^2P_2}{4m} + \frac{C_1}{m}\ln(r) + C_2\right) r\,dr\,d\theta \tag{24}$$

The θ integral is trivial, but the inner integral requires more attention. However, this can be evaluated with many techniques or online tools to see the result will be Eq. (25) below.

$$\bar{v}_z = 2P_2\left[\frac{R^2(1-\kappa)}{8m} + R^2\left(1+\kappa^2\right)\frac{1}{16m} - \frac{R^2}{8m}\left(1 + \frac{(1-\kappa^2)\ln(R)}{\ln(\kappa)}\right)\right] \tag{25}$$

Finding the Volumetric flow rate from here is trivial, as we just have to multiply the average velocity by the area. To make these equations smaller, we will define some variables. First, we will define a parameter A for the cross-sectional area:

$$A = \pi R^2(1-\kappa^2) \tag{26}$$

Next we will define a Parameter B to contain the large term in the brackets in Eq. (25):

$$B = \left[\frac{R^2(1-\kappa)}{8m} + R^2\left(1+\kappa^2\right)\frac{1}{16m} - \frac{R^2}{8m}\left(1 + \frac{(1-\kappa^2)\ln(R)}{\ln(\kappa)}\right)\right] \tag{27}$$

Putting this together, we get the volumetric flow rate as:

$$\dot{V} = 2ABP_2 \tag{28}$$

To adjust this equation for a pipe of finite length, we first need to readjust our definition of pressure. Recall that we said $P_2 = \frac{\partial p}{\partial Z} - \rho g_z$. We will now define $P_3 = p - \rho g_z z$. This is essentially a pressure term that accounts for both the static and the hydrostatic pressure at once. We can then notice $P_e = \frac{\partial P_3}{\partial z}$. Substituting this into our volumetric flow rate equation, we arrive at the first step for our finite pipe equations.

$$\dot{V} = 2AB\frac{\partial P_3}{\partial z} \tag{29}$$

From here, we need to isolate our derivative and integrate over the entire length L and rearrange to arrive at the finite length equation:

$$\frac{\dot{V}}{2AB} = \frac{\partial P_3}{\partial z}$$

Although our pressure is only a function of z, we know that the volume flow rate is constant and therefore the pressure gradient is constant. This means we get the following relationship:

$$\frac{\dot{V}L}{2AB} = P_{3,o} - P_{3,i}$$

Rearranging and dropping the 3 subscript and assuming it is applied for this equation gives us:

$$\frac{P_o - P_i}{L} = \frac{\dot{V}}{2AB} \tag{30}$$

This completes our analysis of a Newtonian Fluid in a Finite Pipe.

Non-Newtonian

Recall Eq. (15) shown below:

$$\frac{\partial v_z}{\partial r} = \left(\frac{r}{2m}\left(\frac{\partial p}{\partial z} - \rho g_z\right) + \frac{C_1}{mr}\right)^{1/n} \tag{15}$$

If we ask Wolfram Alpha to solve this problem, we get the following general solution:

$$v_z = \frac{nr\left(\frac{P_2 r}{2m} + \frac{C_1}{mr}\right)^{1/n}\left(\frac{P_2 r^2}{2C_1} + 1\right)^{-1/n} {}_2F_1\left(-\frac{1}{n}, \frac{n-1}{2n}; \frac{3}{2} - \frac{1}{2n}; -\frac{P_2 r^2}{2C_1}\right)}{n-1} + C_2 \tag{31}$$

In this equation, ${}_2F_1$ represents the Hypergeoemtric function. If we define our domain to ensure $r < 1$, then we can say:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{32}$$

In this equations, the subscripts on a, b, c represent the rising Pochhammer symbol defined as follows:

$$(q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)...(q+n-1) & n > 0 \end{cases} \tag{33}$$

Given this, we can see that this does not have a clean closed form solution that can be exactly valued. The use of computers would be necessary, and we did not have time to implement this function in time for this assignment. Therefore, we call this unsolvable within the scope of this class.

1 Appendix

Below is the code used in the analysis of thIS aasignment

1.1 Plotting the Velocity Profile

```
clear; clc; close all;

% Define the constants m, kappa, and R
m = 0.1;
kappa = 0.3;
R = 1;

% Generate values of P2
P2_values = [-10, -5, -2, -1]; % Add two intermediate values between 0 and -10 if needed

% Define the range of r
lb = kappa * R;
ub = 1;
N = 20;
r = linspace(lb, ub, N);

% Initialize a cell array to store Vz for different P2 values
Vz_values = cell(size(P2_values));

% Initialize variables to store max velocity and its location
max_velocity = -Inf;
max_location = NaN;

% Loop through each value of P2
for i = 1:length(P2_values)
    % Define the value of C1 and C2 for the current P2
    C1 = ((R^2) * P2_values(i) * (1 - kappa^2)) / (4 * log(kappa));
    C2 = -((R^2) * P2_values(i)) / (4 * m) * (1 + ((1 - kappa^2) * log(R)) / log(kappa));

    % Calculate Vz for the current P2
    Vz_values{i} = (r.^2 * P2_values(i)) / (4 * m) + (C1 * log(r)) / m + C2;

    % Find max velocity and its location
    [max_v, idx] = max(Vz_values{i});
    if max_v > max_velocity
        max_velocity = max_v;
        max_location = r(idx);
    end

    % Plot Vz for the current P2
    plot(r, Vz_values{i}, 'DisplayName', sprintf('P_2 = %d', P2_values(i)), 'LineWidth', 2);
    hold on;
end

% Add labels and legend
xlabel('$r$ [m]', 'Interpreter', 'latex', 'FontSize', 14);
ylabel('$V_z$ [m/s]', 'Interpreter', 'latex', 'FontSize', 14);
```

```matlab
title('Velocity Profile', 'Interpreter', 'latex', 'FontSize', 16);
legend();

% Annotate the graph with max velocity and its location
text(max_location, max_velocity, sprintf('Max Velocity: %.2f\nLocation: %.2f', max_velocity, max_location), ...
    'HorizontalAlignment', 'left', 'VerticalAlignment', 'top', 'Interpreter', 'latex');

grid on;
hold off;
set(gcf, 'Color', 'w'); % Set current figure background color to white
```

1.2 Plotting the Shear Stress Profile

```matlab
clear; clc; close all;

% Define the constants m, kappa, and R
m = 0.1;
kappa = 0.3;
R = 1;

% Generate values of P2
P2_values = [-10, -5, -2, -1]; % Add two intermediate values between 0 and -10 if needed

% Define the range of r
lb = kappa * R;
ub = 1;
N = 20;
r = linspace(lb, ub, N);

% Initialize a cell array to store Vz for different P2 values
tau_rz_values = cell(size(P2_values));

% Initialize variables to store max velocity and its location
max_velocity = -Inf;
max_location = NaN;

% Loop through each value of P2
for i = 1:length(P2_values)
    % Define the value of C1 and C2 for the current P2
    C1 = ((R^2) * P2_values(i) * (1 - kappa^2)) / (4 * log(kappa));

   % Calculate tau_rz for the current P2
    tau_rz_values{i} = (r * P2_values(i)) / 2 + C1 ./ r;

    % Plot Vz for the current P2
    plot(r, tau_rz_values{i}, 'DisplayName', sprintf('P_2 = %d', P2_values(i)), 'LineWidth', 2);
    hold on;
end

% Add labels and legend
```

```
xlabel('$r$ [m]', 'Interpreter', 'latex', 'FontSize', 14);
ylabel('$\tau_{rz}$ [Pa]', 'Interpreter', 'latex', 'FontSize', 14);
title('Stress Profile', 'Interpreter', 'latex', 'FontSize', 16);
legend();


grid on;
hold off;
set(gcf, 'Color', 'w'); % Set current figure background color to white
```